

ALMOST-HERMITIAN RANDOM MATRICES: APPLICATIONS TO THE THEORY OF QUANTUM CHAOTIC SCATTERING AND BEYOND*

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It is generally accepted that statistics of energy levels in *closed* chaotic quantum systems is adequately described by the theory of Random Hermitian Matrices. Much less is known about properties of "resonances" - generic features of open quantum systems pertinent for understanding of scattering experiments.

In the framework of the Heidelberg approach to quantum chaotic scattering open systems are characterized by an effective *non-Hermitian* random matrix Hamiltonian \hat{H}_{ef} . Complex eigenvalues of \hat{H}_{ef} are S-matrix poles (resonances). We show how to find the mean density of these poles (Fyodorov and Sommers) and how to use the effective Hamiltonian to calculate autocorrelations of the photodissociation cross section (Fyodorov and Alhassid).

In the second part of the paper we review recent results (Fyodorov, Khoruzhenko and Sommers) on non-Hermitian matrices with independent entries in the regime of *weak Non-Hermiticity*. This regime describes a crossover from Hermitian matrices characterized by Wigner-Dyson statistics of real eigenvalues to strongly non-Hermitian ones whose complex eigenvalues were studied by Ginibre.

I. INTRODUCTION

As is well-known, statistics of highly excited bound states of *closed* quantum chaotic systems of quite different microscopic nature is universal. Namely, it turns out to be independent of the microscopic details when sampled on the energy intervals large in comparison with the mean level separation, but smaller than the energy scale related by the Heisenberg uncertainty principle to the relaxation time necessary for the classically chaotic system to reach equilibrium in the phase space [1]. Moreover, the spectral correlation functions turn out to be exactly those which are provided by the theory of large random matrices on the *local* scale determined by the typical separation $\Delta = \langle X_i - X_{i-1} \rangle$ between neighboring eigenvalues situated around a point X , with brackets standing for the statistical averaging [2]. Microscopic justifications of the use of random matrices for describing the universal properties of quantum chaotic systems have been provided recently by several groups, based both on traditional semiclassical periodic orbit expansions [3,4] and on advanced field-theoretical methods [5,6]. These facts make the theory of random Hermitian matrices a powerful and versatile tool of research in different branches of modern theoretical physics, see e.g. [2,7].

Very recently complex eigenvalues of non-Hermitian random matrices have also attracted much research interest due to their relevance to several branches of theoretical physics. Most obvious motivation comes from the quantum description of *open* systems [8–10] whose fragments can escape, at a given energy, to infinity or come from infinity. For systems of this kind the notion of discrete energy levels loses its validity. Actually, chaotic scattering manifests itself in terms of a high density of poles of the scattering matrix placed irregularly in the complex energy plane. Each of these poles, or *resonances*, $E_k = \mathcal{E}_k - \frac{i}{2}\Gamma_k$, is characterized not only by energy \mathcal{E}_k but also by a finite width Γ_k defined as the imaginary part of the corresponding complex energy and reflecting the finite lifetime of the states in the open system. Recently, the progress in numerical techniques and computational facilities made available high accuracy patterns of resonance poles for realistic atomic and molecular systems in the regime of quantum chaos, see e.g. [12–14].

Due to the presence of these resonances, elements of the scattering matrix show irregular fluctuations as functions of the energy of incoming waves, see [15] and references therein. The main goal of the theory of quantum chaotic scattering is to provide an adequate statistical description of such a behavior.

Whereas the issue of energy level statistics in closed chaotic systems was addressed in an enormous amount of papers statistical characteristics of resonances are much less studied and attracted significant attention only recently, see [8–11] and references in [16].

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One possible way of doing this is to address resonances in the so-called "Heidelberg approach" suggested in the pioneering paper [17] and described in much detail in [16]. The approach turns out to be the most natural framework for incorporating random matrix description of the chaotic scattering.

The starting point of this approach is a division of the Hilbert space of the scattering system into two parts: the "interaction region" and the "channel region". The channel region is supposed to describe a situation of two fragments being apart far enough to neglect any interaction between them. Under these conditions their motion along the collision coordinate is described by a superposition of incoming and outgoing plane waves with wavevectors depending on the internal quantum states of the fragments. We assume that at given energy E exactly M different quantum states of the fragments are allowed, defining M "scattering channels" numbered by the index a .

At the same time, the second part of the Hilbert space is to describe the situation when fragments are close to one another and interact strongly. Correspondingly, any wavefunction of the system $|\Phi(E)\rangle$ can be represented as two-component vector: $|\Phi(E)\rangle = \begin{pmatrix} \mathbf{u} \\ \psi \end{pmatrix}$, with \mathbf{u} and ψ describing the components of the wave function inside the interaction (respectively, channel) region.

Using the standard methods of the scattering theory exposed in detail in [16] one can relate two parts of the wavefunction to one another and finally arrive at the following representation of the energy-dependent scattering matrix \hat{S} in terms of an effective non-Hermitian Hamiltonian $\mathcal{H}_{ef} = \hat{H} - i\hat{\Gamma}$:

$$S_{ab}(E) = \delta_{ab} - 2i\pi \sum_{ij} W_{ai}[E - \mathcal{H}_{ef}]_{ij}^{-1} W_{jb} \quad (1)$$

with the Hermitian Hamiltonian \hat{H} describing the *closed* counterpart of the open system (i.e. interaction region decoupled from the channel one) and the anti-Hermitian part $\hat{\Gamma}$ arising due to a coupling to open scattering channels. In this expression the Hamiltonian \hat{H} is written in some arbitrary basis of states $|i\rangle$, such that $H_{ij} = \langle i|\hat{H}|j\rangle$. The amplitudes W_{ai} , $a = 1, 2, \dots, M$ are matrix elements coupling the internal motion in an "internal" state $|i\rangle$ to one out of M open channels a . One also has to choose the anti-Hermitian part to be $\hat{\Gamma} = \pi \sum_a W_{ia} W_{ja}^*$ in order to ensure the unitarity of the $M \times M$ scattering matrix $\hat{S}(E)$ [18–20].

It is natural to expect, that universal properties of open chaotic systems are inherited from the corresponding universality of levels of their closed counterparts. Of course, one can expect a relation of this kind only when incoming particles stay inside the interaction region long enough to be able to experience the chaoticity of internal dynamics. Going from the time domain to the energy domain, this fact suggests that only scattering characteristics on a scale shorter than inverse classical relaxation time on the energy shell are expected to be universal. Another characteristic energy scale in this domain is a typical level spacing Δ of the closed counterpart of our quantum open system. Thus, we expect the scattering characteristics (in particular, the statistics of resonances) to be universal on the scale comparable with Δ . In contrast, smooth energy dependence of S -matrix elements on a much larger energy scale must be system-specific.

The next step is to incorporate the random matrix description of quantum chaotic systems by replacing the Hamiltonian \hat{H} by a random matrix of appropriate symmetry. Namely, chaotic systems with preserved time-reversal invariance (TRI) should be described by matrices H_{ij} which are real symmetric. Such matrices form the Gaussian Orthogonal Ensemble, whereas for systems with broken TRI one uses complex Hermitian matrices from the Gaussian Unitary Ensemble [2,7].

The third essential ingredient of the Heidelberg approach is performing the ensemble averaging non-perturbatively in the framework of the so-called supersymmetry technique. It was invented initially by Efetov in the context of theory of disordered metals and the Anderson localization [21,22] and adjusted for the description of quantum chaotic scattering by Verbaarschot, Weidenmüller and Zirnbauer [17].

The Heidelberg approach turns out to be a very powerful tool for extracting universal properties of open chaotic systems. In the first part of the paper we outline the derivation of the mean resonance density in the complex plane following Fyodorov and Sommers [9]. Another recent example of the utility of the non-Hermitian effective Hamiltonian $\hat{H} - i\hat{\Gamma}$ is that its resolvent is related to the probability for an excited system to decay via open channels. For this reason it can be used to calculate the statistics of such quantities as e.g. photodissociation cross-section in the regime of quantum chaos. We present the formula for cross-section autocorrelation function derived recently by Fyodorov and Alhassid [23].

The fact that non-selfadjoint operators appear quite generally when one considers open systems of various types is known for a long time [20]. It is therefore not surprising that open quantum systems were the first examples of applications of non-Hermitian random matrices [8,9,16], see also recent papers [10,24,25]. Other early applications included also studies of dissipative quantum maps [26–28] and chaotic dynamics of asymmetric neural networks [29].

Recently, however, random matrices (more generally, random linear operators) with complex eigenvalues emerged in many other physical contexts. Let us briefly mention the most interesting examples.

- The effective Hamiltonian describing a thermal motion of an isolated vortex in disordered type-II superconductors with columnar defects has a form of that for a quantum particle in an *imaginary* vector potential \mathbf{A} :

$$\hat{H} = \frac{1}{2m} (-i\hbar\nabla + i\mathbf{A})^2 + V(\mathbf{r}), \quad (2)$$

with $V(\mathbf{r})$ being a random potential generated by defects. The imaginary vector potential makes the Hamiltonian to be a non-Hermitian one. This fact pointed out by Hatano and Nelson [30] gave a boost to several interesting studies [31,32].

- A classical diffusing particle advected by a stationary random velocity field \mathbf{v} is described by a non-Hermitian Fokker-Plank random operator \mathcal{L}_{FP} [33]:

$$\frac{\partial}{\partial t} n(\mathbf{r}, t) = \mathcal{L}_{FP} n(\mathbf{r}, t) = (D\nabla^2 - \nabla\mathbf{v}) n(\mathbf{r}, t) \quad (3)$$

where $n(\mathbf{r}, t)$ is the density of particles and D is the diffusion constant.

- Recent attempts to understand the universal features of chiral symmetry breaking in Quantum Chromodynamics required to consider quarks in a finite chemical potential μ interacting with the Yang-Mills gauge field. The corresponding partition function is given by:

$$\mathcal{Z}(m, \mu) = \left\langle \prod_{f=1}^{N_f} \det \left(m_f + \mu\hat{\gamma}_0 + \hat{\mathcal{D}} \right) \right\rangle_{\mathcal{A}} \quad (4)$$

where $\hat{\mathcal{D}} = \hat{\gamma}_\mu \partial_\mu + i\hat{\gamma}_\mu \mathcal{A}_\mu$ is the Euclidean Dirac operator in the gauge vector potential \mathcal{A} , and $\langle \dots \rangle_{\mathcal{A}} = \int D\mathcal{A}(\dots) \exp - \int d^4x F_{\mu\nu}^2$ stands for the averaging over all configurations of the gauge field $F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$

A finite chemical potential μ makes the corresponding operator to be a non-selfadjoint one with complex eigenvalues. This fact makes a problem of numerical evaluation of the partition function by lattice simulations to be a very difficult one. Recently it was suggested, that some universal features of the model can be correctly recovered if one replaces the true gauge-field averaging by averaging over random gauge-field configurations. As a result, one comes to a class of non-Hermitian random matrix problems of a particular type [34].

- Finally, let us mention that there exist several interesting links between complex eigenvalues of non-Hermitian random matrices and systems of interacting particles in one and two spatial dimensions [35,36].

At the same time, our knowledge of statistical properties of random non-selfadjoint matrices is quite scarce and incomplete. This fact recently stimulated efforts of different groups to improve our understanding in this direction [37]- [42].

Traditional mathematical treatment of random matrices with no symmetry conditions imposed goes back to the pioneering work by Ginibre [43] who determined all the correlation functions of the eigenvalues in an ensemble of complex matrices with independent Gaussian entries. The progress in the field was rather slow but steady, see [44,45,29,46-48,40,49].

Surprisingly, all these studies completely disregarded the existence of a nontrivial regime of *weak non-Hermiticity* recognized in the work by Fyodorov, Khoruzhenko and Sommers [38], see more detailed discussion in [50]. The guiding idea to realize the existence of such a regime comes from the experience with resonances [9]. Guided by that example one guesses that a new regime occurs when the imaginary part of typical eigenvalues is comparable with a mean *separation* between neighboring eigenvalues along the real axis.

One can again employ the same supersymmetry approach which was used to study resonances and obtain the mean density of complex eigenvalues in the regime of weak non-Hermiticity for matrices with independent elements [38,50]. The density turned out to be described by a formula containing as two opposite limiting cases both the Wigner semicircular density of real eigenvalues typical for Hermitian random matrices and the uniform density of complex eigenvalues discovered for strongly non-Hermitian random matrices already by Ginibre [43], in much details addressed by Girko [45] and studied for different cases by other authors [29,48,49].

Very recently, Efetov [32] showed the relevance of almost-Hermitian random matrices to the very interesting problem of motion of flux lines in superconductors with columnar defects [30]. He also managed to derive the density of complex eigenvalues for a related, but different set of almost-symmetric real random matrices. This development clearly shows that, apart from being a rich and largely unexplored mathematical object, almost-Hermitian random matrices enjoy direct physical applications and deserve to be studied in more detail.

Actually, the non-Hermitian matrices considered in [38] and [32] are just two limiting cases of a general three-parameter family of non-Hermitian ensembles [50]. In second section of the paper we outline the derivation of this fact and present the resulting expression in terms of a non-linear σ -model integral.

Although giving an important insight into the problem, the supersymmetry non-linear σ -model technique suffers from at least two deficiencies. The most essential one is that the present state of art in the application of the supersymmetry technique gives little hope of access to quantities describing *correlations* between different eigenvalues in the complex plane due to insurmountable technical difficulties. At the same time, conventional theory of random Hermitian matrices suggests that these *universal* correlations are the most interesting features. The second drawback is less important for a physicist, but a crucial one for the mathematicians: at the moment the supersymmetry technique can not be considered as a rigorous mathematical tool and has the status of a powerful heuristic method.

Fortunately, for the simplest case of almost-Hermitian Gaussian random matrices one can develop the rigorous mathematical theory based on the method of orthogonal polynomials. Such a method is free from the above mentioned problem and allows one to study correlation properties of complex spectra to the same degree as is typical for earlier studied classes of random matrices [39]. We briefly discuss the obtained results in the end of the paper. The detailed exposition of the method and the derivation of the results can be found in [50]. Unfortunately, the paper [39] contained a number of misleading misprints. For this reason we indicate those misprints in the present text by using footnotes.

II. NON-HERMITIAN RANDOM MATRICES IN THE THEORY OF CHAOTIC QUANTUM SCATTERING

To calculate the density of resonance poles in the complex energy plane we notice that they are just eigenvalues of the effective non-Hermitian Hamiltonian introduced in Eq.(1).

Without any loss of generality coupling amplitudes W_{aj} can be chosen in a way ensuring that the average S -matrix is diagonal in the channel basis: $\overline{S_{ab}} = \delta_{ab} \overline{S_{aa}}$. Then one finds the following expression [17]:

$$\overline{S_{aa}} = \frac{1 - \gamma_a g(X)}{1 + \gamma_c g(X)}; \quad \gamma_a = \pi \sum_i W_{ia}^* W_{ia} \quad (5)$$

where $g(X) = iX/2 + \pi \overline{\nu(X)}$ and $\overline{\nu(X)} = (1 - X^2/4)^{1/2}$ is the semicircular level density. The strength of coupling to continua is convenient to characterize via the transmission coefficients $T_a = 1 - |\overline{S_{aa}}|^2$. These quantities measure a part of the incoming flux in a given channel that spends a substantial part of the time in the interaction region [17,51]. The case $T_a \ll 1$ corresponds to almost closed channel a , whereas the opposite limiting case $T_a = 1$ corresponds to the perfect coupling between the interaction region and the channel a . It is easy to see that both limits $\gamma_a \rightarrow 0$ and $\gamma_a \rightarrow \infty$ equally correspond to the weak effective coupling regime whereas the strongest coupling (at fixed energy X) corresponds to $\gamma_a = 1$.

In the case of weak effective coupling to continua individual resonances do not overlap: $\langle \Gamma \rangle \ll \Delta$, with $\Delta = (\nu(X)N)^{-1}$ standing for the mean level spacing of the "closed" system and $\langle \Gamma \rangle$ standing for the mean resonance width. Under these conditions one can use a simple first order perturbation theory to calculate resonance widths in terms of eigenfunctions of the closed system. Since different components of eigenvectors of large random matrices are decorrelated and Gaussian-distributed, one finds in such a procedure that the scaled widths $y_s = \frac{\Gamma}{\langle \Gamma \rangle}$ are distributed according to the so-called χ^2 -distribution:

$$\rho(y_s) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} y_s^{\nu/2-1} e^{-\frac{\nu}{2} y_s} \quad (6)$$

where $\Gamma(z)$ stands for the Gamma function and the parameter $\nu = M$ ($\nu = 2M$) for systems with preserved (broken) time reversal invariance, M being the number of open scattering channels. The case $\nu = 1$ is known as Porter-Thomas distribution and was shown to be in agreement with experimental data (see some references in [52]).

Experimentally, one quite frequently encounter the case of only $M \sim 1$ open channels and $\langle \Gamma \rangle \sim \Delta$, see e.g. [53–55]. Under this situation resonances overlap considerably and one can not use perturbation theory any longer. The problem of determining the statistical characteristics of the chaotic scattering in the regime of (partly or completely)

overlapping resonances is of essentially non-perturbative nature. As a result, one has to use some non-perturbative methods allowing to evaluate the ensemble averaging exactly for arbitrary ratio $\langle \Gamma \rangle / \Delta$.

Fortunately, one can study very efficiently various universal statistical features of chaotic quantum scattering by performing the ensemble averaging with the use of the supersymmetry method [17].

One can recover the spectral density

$$\rho(Z) = \sum_{k=1}^N \delta^{(2)}(Z - Z_k) = \sum_{k=1}^N \delta(X - X_k) \delta(Y - Y_k) \equiv \rho(X, Y) \quad (7)$$

of complex eigenvalues $Z_k = X_k + iY_k$, $k = 1, 2, \dots, N$ if one knows the "potential" [29]:

$$\Phi(X, Y, \kappa) = \frac{1}{2\pi} \ln \text{Det}[(Z - \mathcal{H}_{ef})(Z - \mathcal{H}_{ef})^\dagger + \kappa^2]$$

in view of the relation: $\rho(X, Y) = \lim_{\kappa \rightarrow 0} \partial^2 \Phi(X, Y, \kappa)$, where ∂^2 stands for the two-dimensional Laplacian. Technically, it is convenient to introduce the generating function (cf. [16])

$$\mathcal{Z} = \frac{\text{Det}[(Z - \mathcal{H}_{ef})(Z - \mathcal{H}_{ef})^\dagger + \kappa^2]}{\text{Det}[(Z_b - \mathcal{H}_{ef})(Z_b - \mathcal{H}_{ef})^\dagger + \kappa^2]} \quad (8)$$

in terms of which

$$\rho(Z) = -\frac{1}{\pi} \lim_{\kappa \rightarrow 0} \frac{\partial}{\partial Z^*} \lim_{Z_b \rightarrow Z} \frac{\partial}{\partial Z_b} \mathcal{Z} \quad (9)$$

To facilitate the ensemble averaging we follow the standard route and represent the ratio of the two determinants

in Eq.(8) in terms of a Gaussian superintegral over eight-component supervectors $\Phi_i = \begin{pmatrix} \Psi_i(+), \\ \Psi_i(-) \end{pmatrix}$ where $\Psi_i(\pm) = \begin{pmatrix} \vec{R}_i(\pm) \\ \vec{\eta}_i(\pm) \end{pmatrix}$ and

$$\vec{R}_i(\pm) = \begin{pmatrix} r_i(\pm) \\ r_i^*(\pm) \end{pmatrix} \quad ; \quad \vec{\eta}_i(\pm) = \begin{pmatrix} \chi_i(\pm) \\ \chi_i^*(\pm) \end{pmatrix} \quad ; \quad \vec{\eta}_i^\dagger(\pm) = (\chi_i^*(\pm); -\chi_i(\pm))$$

with components $r_i(+), r_i(-)$; $i = 1, 2, \dots, N$ being complex commuting variables and $\chi_i(+), \chi_i(-)$ forming the corresponding Grassmann parts of the supervectors $\Psi_i(\pm)$.

Further evaluation goes along the lines discussed in [16] in more detail. After a set of standard manipulations one arrives at the following expression for the density $\rho_X(y) \equiv \frac{1}{\pi} \rho(X, Y) \Delta^2(X)$ of scaled resonance widths $y = \frac{\pi \Gamma}{\Delta}$ (measured in units of the local mean level spacing $\Delta(X)$ of the closed system) for the resonances whose positions are within a narrow window around the point X of the spectrum:

$$\langle \rho_X(y) \rangle = \frac{1}{16} \int d\mu(\hat{Q}) \text{Str}(\hat{\sigma}_\tau^{(F)} \hat{Q}) \text{Str}(\hat{\sigma}_\tau \hat{Q}) \exp \frac{i}{4} y \text{Str}(\hat{\sigma}_\tau \hat{Q}) \prod_{a=1}^M \text{Sdet}^{-1/4} \left[1 - \frac{i}{2g_a} \{ \hat{Q}, \hat{\sigma}_\tau \} \right] \quad (10)$$

Here the integration goes over the set of 8×8 supermatrices \hat{Q} satisfying the constraint $\hat{Q}^2 = -1$ and $\{ \hat{Q}, \hat{\sigma}_\tau \} = \hat{Q} \hat{\sigma}_\tau + \hat{\sigma}_\tau \hat{Q}$ stands for the anticommutator. Properties of these matrices and the integration measure $d\mu(\hat{Q})$ can be found in [21]. Other 8×8 supermatrices entering the expression Eq.(10) are as follows:

$$\hat{\sigma}_\tau^{(F)} = \begin{pmatrix} \hat{0}_4 & \hat{\tau}_3^{(F)} \\ \hat{\tau}_3^{(F)} & \hat{0}_4 \end{pmatrix}; \quad \hat{\sigma}_\tau = \begin{pmatrix} 0 & \hat{\tau}_3 \\ \hat{\tau}_3 & 0 \end{pmatrix}$$

and $\hat{\tau}_3; \hat{\tau}_3^{(F)}$ are 4×4 diagonal supermatrices: $\hat{\tau}_3 = \text{diag}\{\hat{\tau}, \hat{\tau}\}$; $\hat{\tau}_3^{(F)} = \text{diag}\{\hat{0}_2, \hat{\tau}\}$. with $\hat{\tau} = \text{diag}(1, -1)$. We also introduced quantities $g_a = \frac{1}{2\pi\bar{\rho}(X)}(\gamma_a + \gamma_a^{-1})$ related to the transmission coefficients as $g_a = 2/T_a - 1$ and used the symbols Str, Sdet for the graded trace and the graded determinant, correspondingly.

The expression above is valid for chaotic systems with preserved as well as with broken time-reversal invariance. To extract the explicit form of the distribution function one still has to perform the integration over the manifold of the supermatrices Q which is different for two cases. In general it is a rather difficult calculation due to a cumbersome

parametrisation of that manifold. At the moment the result is known for the simplest case of the systems with broken time-reversal invariance [9,16]. For the sake of simplicity we present this distribution for the case of equivalent channels $a = 1, \dots, M$ with equal transmission coefficients $T_a = T$.

First of all, it turns out that the mean resonance width is related to the transmission coefficient T as:

$$\langle \Gamma \rangle = -\Delta \frac{M}{2\pi} \ln(1-T) \equiv \Delta \frac{M}{2\pi} \ln \frac{g+1}{g-1} \quad (11)$$

The formula Eq.(11) is well known in nuclear physics as Moldauer-Simonius relation [56].

It is convenient to use the parameter $\kappa = -\frac{M}{2} \ln(1-T)$ as a measure of the resonance overlap. It is related to the mean widths as $\kappa = \pi \frac{\langle \Gamma \rangle}{\Delta}$ and therefore gives a typical number of neighboring resonances that overlap substantially.

Measuring the resonance widths in units of the mean widths $\langle \Gamma \rangle$ one finds the following distribution function.

$$\begin{aligned} \rho\left(y_s = \frac{\Gamma}{\langle \Gamma \rangle}\right) &= \frac{1}{2\Gamma(M)\kappa y^2} \int_{y_s b_-}^{y_s b_+} dt t^M e^{-t} \\ &= (-1)^M \frac{y_s^{M-1}}{\Gamma(M)} \frac{d^M}{dy_s^M} \left(\exp -[\kappa \coth(\kappa/M) y_s] \frac{\sinh \kappa y_s}{\kappa y_s} \right). \end{aligned} \quad (12)$$

where we used the notations $b_{\pm} = \kappa e^{\pm \kappa/M} / \sinh(\kappa/M)$ and $\Gamma(M) = (M-1)!$ for the Euler gamma-function..

Properties of this distribution are discussed in much details in [16], also for the case of non-equivalent channels. Let us only briefly mention the most interesting features.

Quick inspection of eq.(12) shows that it is indeed reduced to the χ^2 distribution, eq.(6) when the effective coupling to continua is weak: $\kappa \ll 1$. Under this condition resonances are typically too narrow to overlap with others: $\Gamma \ll \Delta$. However, as long as the effective coupling becomes stronger: $T \rightarrow 1$, hence $g \gg 1$ the parameter κ grows large. Under these conditions another domain of resonance widths becomes more and more important:

$$e^{-\kappa/M} \frac{\sinh(\kappa/M)}{(\kappa/M)} < y_s < e^{\kappa/M} \frac{\sinh(\kappa/M)}{(\kappa/M)},$$

where the distribution eq.(12) shows the powerlaw decrease: $\rho(y_s) \approx \frac{1}{2\kappa} M y_s^{-2}$. The most drastic difference from eq.(6) occurs for the maximal effective coupling $g = 1$ (i.e $\kappa = \infty$). In this regime the powerlaw tail extends up to infinity, making all positive moments (starting from the first one) to be apparently divergent. One can argue that the powerlaw tail $M y_s^{-2}$ turns out to be dictated by classical processes of exponential escape typical for fully chaotic systems [16]. The rate of this escape in the semiclassical limit $M \gg 1$ is provided by the value of the gap in the distribution of resonance width, see [8] for a more detailed discussion.

The best candidates for checking the applicability of eq.(12) to real physical systems are realistic models of ballistic mesoscopic devices subject to an applied magnetic field that serves to break the TRI [53]. It is however quite clear that all the basic qualitative features of the distribution eq.(12) (in particular, the powerlaw behavior $\rho(y_s) \propto M y_s^{-2}$ for the overlapping resonance regime) should be valid for the systems with preserved TRI as well. Recent numerical data [57] support the validity of this conjecture.

We have seen, that the non-Hermitian random matrix Hamiltonian \hat{H}_{ef} appeared naturally in the scattering matrix description of open quantum systems. Actually, such a Hamiltonian is the most adequate tool to describe the quantum relaxation processes such as escape of the particle from the interaction region in the regime of quantum chaos. Some aspects of such a relaxation were studied some time ago in [58] and reconsidered in more details recently by Savin and Sokolov [24] who used insights provided by the resonance widths distribution Eq.(12)

It is therefore quite natural that the resolvent of the non-Hermitian effective Hamiltonian $\hat{H}_{ef} = \hat{H} - i\hat{\Gamma}$ is related to the probability for an excited system to decay via one of open channels. For this reason it can be used to calculate the statistics of such quantities as the Wigner *time delay* [59] which is a measure of mean time spent by a scattered particle inside the interaction region.

One more example of the utility of the resolvent of the non-Hermitian effective Hamiltonian \hat{H}_{ef} is that it can be related to the total photodissociation crosssection in the regime of quantum chaos. The formula for the cross section autocorrelation function was derived recently by Fyodorov and Alhassid [23]. Below we outline the derivation and present the final result.

The total energy-dependent cross section $\sigma(E)$ is defined as a probability to be excited from a ground state $|g\rangle$ and to dissociate at a given energy E per unit time and per unit incoming photon flux density. In the dipole approximation it is given by (see, e.g. detailed discussion in [60]):

$$\sigma(E) = \sigma_0 \sum_a \left| \langle g | \hat{\mu} | \Phi_a^{out}(E) \rangle \right|^2 \quad (13)$$

Here $\hat{\mu}$ is the dipole operator $\hat{\mu} = e\mathcal{E}\mathbf{r}$ of the system in the external electric field \mathcal{E} , σ_0 is a constant proportional to the excitation energy E and $|\Phi_a^{out}(E)\rangle$ is the exact wave function of the system at energy E subject to *outgoing* boundary conditions in one of the open channels $a = 1, 2, \dots, M$.

It turns out, that for many systems of interest (e.g. molecules HO_2 [54] and H_3^+ [14]), the cross section patterns look like irregular fluctuating signals consisting of many randomly positioned (partly) overlapping resonance peaks. Such a behavior (typical also for the resonance scattering in atomic systems [12,55]) has its origin in the underlying pattern of resonances in the complex plane.

We already mentioned that one can relate the "internal" and "external" parts of the wavefunctions by using the resolvent of the non-Hermitian effective Hamiltonian \hat{H}_{ef} . In the present case such a relation can be written as (see e.g. [61]):

$$\mathbf{u} = \left(E - \hat{H}_{ef}^\dagger\right)^{-1} \hat{W}\mathbf{B} \quad (14)$$

where M -component vector \mathbf{B} contains amplitudes of outgoing waves in each of the open channels.

The ground state wavefunction describes a bound state and as such has no components outside the interaction region. Using this fact and Eq.(14) one finds after some algebraic manipulations that the cross section Eq.(13) can be rewritten in the following convenient form:

$$\sigma(E) \propto \text{Im} \left\langle g \left| \mu \frac{1}{E - \mathcal{H}_{ef}} \mu \right| g \right\rangle \quad (15)$$

which is just a particular case of the optical theorem. One also can arrive at the expression Eq.(15) by resumming the perturbation theory, see [55] for more details and relevant references.

The advantage of the form Eq.(15) is that it expresses the photodissociation cross section in terms of the resolvent of an effective non-Hermitian operator $\mathcal{H}_{ef} = H_{in} - i\pi WW^\dagger$ which is known to describe open chaotic systems in the random matrix formalism. It allows to apply very well developed methods of evaluating averages of products of resolvents based on the Efetov supermatrix formalism. Measuring energy separations in units of the mean level spacing of the closed system Δ one finds in such a calculation the cross section autocorrelation function [23]:

$$S(\omega = 2\pi\Omega/\Delta) = \frac{\langle \sigma(E - \Omega/2) \sigma(E + \Omega/2) \rangle}{\langle \sigma(E) \rangle^2} - 1,$$

to be a sum of two terms $S(\omega) = S_1(\omega) + S_2(\omega)$ given by the following expressions:

$$S_{1,2}(\omega) = \int_{-1}^1 d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \frac{\cos[\omega(\lambda_1\lambda_2 - \lambda)](1 - \lambda^2)}{[\lambda_1^2 + \lambda_2^2 + \lambda^2 - 2\lambda_1\lambda_2\lambda - 1]^2} f_{1,2}(\lambda, \lambda_1, \lambda_2) \prod_{c=1}^M \frac{(g_a + \lambda)}{[(g_a + \lambda_1\lambda_2)^2 - (\lambda_1^2 - 1)(\lambda_2^2 - 1)]^{1/2}} \quad (16)$$

where

$$f_1(\lambda, \lambda_1, \lambda_2) = (\lambda_1\lambda_2 - \lambda)^2; \quad f_2(\lambda, \lambda_1, \lambda_2) = 2\lambda_1^2\lambda_2^2 - \lambda_1^2 - \lambda_2^2 - \lambda^2 + 1$$

The parameters g_a were introduced before and related to the transmission coefficients as $g_a = 2/T_a - 1$.

It is worth mentioning that each of the contributions $S(\omega)_{1,2}$ represent an interesting object by itself. Namely, $S_1(\omega)$ coincides with the autocorrelation function of the so-called Wigner time delays studied in some detail in [59], whereas $S_2(\omega)$ is related by the Fourier-transform to the so-called "norm leakage" out the interaction region. The latter quantity was introduced recently by Savin and Sokolov as a characteristic of the process of quantum relaxation in chaotic systems and studied in detail for the simplest case of broken time-reversal invariance [24].

Actually, the starting Fermi golden rule formula Eq.(13) is valid for an arbitrary excitation of the system with a weak perturbation $\hat{\mu}$. For this reason the autocorrelation function of crossections presented above is also of a general applicability.

Finally, it is necessary to mention that in the limit $T_a = 0$ for all a (corresponding to a closed system with purely bound spectra and no possibility for photodissociation) the expression Eq.(16) reduces to the "oscillator strength" correlation function found by Taniguchi et al. [62].

III. NON-HERMITIAN MATRICES WITH INDEPENDENT ELEMENTS: UNIVERSAL PROPERTIES IN THE REGIME OF WEAK NON-HERMITICITY

To begin with, any $N \times N$ matrix \hat{J} can be decomposed into a sum of its Hermitian and skew-Hermitian parts: $\hat{J} = \hat{H}_1 + i\hat{H}_2$, where $\hat{H}_1 = (\hat{J} + \hat{J}^\dagger)/2$ and $\hat{H}_2 = (\hat{J} - \hat{J}^\dagger)/2i$. Following this, we consider an ensemble of random $N \times N$ complex matrices $\hat{J} = \hat{H}_1 + i\hat{H}_2$ where \hat{H}_p ; $p = 1, 2$ are both Hermitian: $\hat{H}_p^\dagger = \hat{H}_p$. The parameter v is used to control the degree of non-Hermiticity.

In turn, complex Hermitian matrices \hat{H}_p can always be represented as $\hat{H}_1 = \hat{S}_1 + iu\hat{A}_1$ and $\hat{H}_2 = \hat{S}_2 + iw\hat{A}_2$, where $\hat{S}_p = \hat{S}_p^T$ is a real symmetric matrix, and $\hat{A}_p = -\hat{A}_p^T$ is a real antisymmetric one. From this point of view the parameters u, w control the degree of being non-symmetric.

Throughout the paper we consider the matrices $\hat{S}_1, \hat{S}_2, \hat{A}_1, \hat{A}_2$ to be mutually statistically independent, with i.i.d. entries normalized in such a way that:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr} \hat{S}_p^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr} \hat{A}_p \hat{A}_p^T = 1 \quad (17)$$

As is well-known [2], this normalisation ensures that for any value of the parameter $u \neq 0$, such that $u = O(1)$ when $N \rightarrow \infty$ statistics of real eigenvalues of the Hermitian matrix of the form $\hat{H} = \hat{S} + iu\hat{A}$ is identical (up to a trivial rescaling) to that of $u = 1$, the latter case known as the Gaussian Unitary Ensemble (GUE). On the other hand, for $u \equiv 0$ real eigenvalues of the real symmetric matrix \hat{S} follow another pattern of the so-called Gaussian Orthogonal Ensemble (GOE).

The non-trivial crossover between GUE and GOE types of statistical behavior happens on a scale $u \propto 1/N^{1/2}$ [63]. This scaling can be easily understood by purely perturbative arguments [64]. Namely, for $u \propto 1/N^{1/2}$ a typical shift $\delta\lambda$ of eigenvalues of the symmetric matrix S due to the antisymmetric perturbation $iu\hat{A}$ is of the same order as the mean spacing Δ between unperturbed eigenvalues: $\delta\lambda \sim \Delta \sim 1/N$.

Similar perturbative arguments show [38], that the most interesting behavior of *complex* eigenvalues of non-Hermitian matrices should be expected for the parameter v being scaled in a similar way: $v \propto 1/N^{1/2}$. It is just the regime when the *imaginary* part $\text{Im}Z_k$ of a typical eigenvalue Z_k due to non-Hermitian perturbation is of the same order as the mean spacing Δ between unperturbed real eigenvalues: $\text{Im}Z_k \sim \Delta \sim 1/N$. Under these conditions a non-Hermitian matrix J still "remembers" the statistics of its Hermitian part \hat{H}_1 . As will be clear afterwards, the parameter w should be kept of the order of unity in order to influence the statistics of the complex eigenvalues.

It is just this regime of *weak non-Hermiticity* which we are interested in. Correspondingly, we scale the parameters as ¹:

$$v = \frac{\alpha}{2\sqrt{N}}; \quad u = \frac{\phi}{2\sqrt{N}} \quad (18)$$

and consider α, ϕ, w fixed of the order $O(1)$ when $N \rightarrow \infty$.

To be specific, we consider the real symmetric matrix \hat{S}_1 to be taken from the ensemble of *sparse* random matrices [65,66] characterized by the following probability density of a given entry S_{ij} :

$$\mathcal{P}(S_{ij}) = (1 - \frac{p}{N})\delta(S_{ij}) + \frac{p}{N}h(S_{ij}) \quad (19)$$

where $h(s) = h(-s)$ is an arbitrary even distribution function satisfying the conditions: $h(0) < \infty$; $\int h(s)s^2 ds < \infty$ and p stands for the mean value of non-zero matrix elements per column. Actually, this ensemble is the most general one among those with independent elements, and statistics of its eigenvalues was proved to be completely universal [65,67], up to a rescaling by *ensemble-dependent* mean eigenvalue density $\nu(X)$. Statistics of the matrix elements of all other matrices $S_2, A_{1,2}$ is immaterial as long as their elements are statistically independent as well.

The calculation of the mean density of complex eigenvalues follows essentially the same route as that outlined in the previous section for the resonances. The method used [50] is a generalization of the Efetov's technique to the case of sparse random matrices suggested in [65] (see some details also in [66]). As the result, one arrives at the following expression [50]:

¹In the Letter [39] there is a misprint in the definition of the parameter α .

$$\langle \rho(X, y) \rangle = \frac{N\nu(X)}{16} \int d\mu(\hat{Q}) \text{Str} \left(\hat{\sigma}_\tau^{(F)} \hat{Q} \right) \text{Str} \left(\hat{\sigma}_\tau \hat{Q} \right) \exp -S(\hat{Q}) \quad (20)$$

$$S(\hat{Q}) = -\frac{i}{2} y \text{Str} \left(\hat{\sigma}_\tau \hat{Q} \right) - \frac{a^2}{16} \text{Str} \left(\hat{\sigma}_\tau \hat{Q} \right)^2 + \frac{b^2}{16} \text{Str} \left(\hat{\tau}_2 \hat{Q} \right)^2 - \frac{c^2}{16} \text{Str} \left(\hat{\sigma} \hat{Q} \right)^2 \quad (21)$$

where we introduced the scaled imaginary parts $y = \pi\nu(X)NY$ and used the notations: $a^2 = (\pi\nu(X)\alpha)^2$, $b^2 = (\pi\nu(X)\phi)^2$, $c^2 = (\pi\nu(X)\alpha w)^2$. The supermatrices $\hat{\tau}_2$ and $\hat{\sigma}$ entering this expressions are as follows:

$$\hat{\tau}_2 = \text{diag}\{\hat{\tau}_3, \hat{\tau}_3\}; \quad \sigma = \begin{pmatrix} 0 & \hat{I}_4 \\ \hat{I}_4 & 0 \end{pmatrix}$$

and the supermatrices $\hat{\sigma}_\tau$ and $\hat{\tau}_3$ were defined after Eq.(10). The expression (20) is just the universal σ - model representation of the mean density of complex eigenvalues in the regime of weak non-Hermiticity we were looking for. The universality is clearly manifest: all the particular details about the ensembles entered only in the form of mean density of real eigenvalues $\nu(X)$. The density of complex eigenvalues turns out to be dependent on three parameters: a, b and c , controlling the degree of non-Hermiticity (a), and symmetry properties of the Hermitian part (b) and non-Hermitian part (c).

The following comment is appropriate here. The derivation above can be done not only for ensembles with i.i.d. entries but also for any "rotationally invariant" ensemble of real symmetric matrices \hat{S}_1 . To do so one can employ the procedure invented by Hackenbroich and Weidenmüller [68] allowing one to map the correlation functions of the invariant ensembles (plus perturbations) to that of Efetov's σ -model.

Still, in order to get an explicit expression for the density of complex eigenvalues one has to evaluate the integral over the set of supermatrices \hat{Q} . In general, it is an elaborate task due to complexity of that manifold.

At the present moment such an evaluation was successfully performed for two important cases: those of almost-Hermitian matrices and real almost-symmetric

matrices. The first case (which is technically the simplest one) corresponds to $\phi \rightarrow \infty$, that is $b \rightarrow \infty$. Under this condition only that part of the matrix \hat{Q} which commutes with $\hat{\tau}_2$ provides a nonvanishing contribution. As the result, $\text{Str} \left(\hat{\sigma} \hat{Q} \right)^2 = \text{Str} \left(\hat{\sigma}_\tau \hat{Q} \right)^2$ so that second and fourth term in Eq.(20) can be combined together. Evaluating the resulting integral, and introducing the notation $\tilde{a}^2 = a^2 + c^2$ one finds [38]:

$$\rho_X(y) = \sqrt{\frac{2}{\pi}} \frac{1}{\tilde{a}} \exp \left(-\frac{2y^2}{\tilde{a}^2} \right) \int_0^1 dt \cosh(2ty) \exp(-\tilde{a}^2 t^2/2), \quad (22)$$

where $\rho_X(y)$ is the density of the scaled imaginary parts y for those eigenvalues, whose real parts are situated around the point X of the spectrum (cf. Eq.(10)).

It is easy to see, that when \tilde{a} is large one can effectively put the upper boundary of integration in Eq.(22) to be infinity due to the Gaussian cut-off of the integrand. This immediately results in the uniform density $\rho_X(y) = (\tilde{a}^2)^{-1}$ inside the interval $|y| < \tilde{a}^2/2$ and zero otherwise. Translating this result to the two-dimensional density of the original variables X, Y , we get:

$$\rho(X, Y) = \begin{cases} \frac{N}{4\pi v^2(1+w^2)} & \text{for } |Y| \leq 2\pi\nu(X)v^2(1+w^2) \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

This result is a natural generalization of the so-called "elliptic law" known for strongly non-Hermitian random matrices [43,45,29]. Indeed, the curve encircling the domain of the uniform eigenvalue density is an ellipse: $\frac{Y^2}{2v^2(1+w^2)} + \frac{X^2}{4} = 1$ as long as the mean eigenvalue density of the Hermitian counterpart is given by the semicircular law. The semicircular density is known to be shared by ensembles with i.i.d. entries, provided the mean number p of non-zero elements per row grows with the matrix size as $p \propto N^\alpha$; $\alpha > 0$, see [65]. In the general case of sparse or "rotationally invariant" ensembles the function $\nu(X)$ might be quite different from the semicircular law. Under these conditions Eq.(23) still provides us with the corresponding density of complex eigenvalues.

The second nontrivial case for which the result is known explicitly is due to Efetov [32]. It is the limit of slightly asymmetric real matrices corresponding in the present notations to: $\phi \rightarrow 0; w \rightarrow \infty$ in such a way that the product $\phi w = \tilde{c}$ is kept fixed. The density of complex eigenvalues turns out to be given by:

$$\rho_X(y) = \delta(y) \int_0^1 dt \exp(-\tilde{c}^2 t^2/2) + 2\sqrt{\frac{2}{\pi}} \frac{|y|}{\tilde{c}} \int_1^\infty du \exp\left(-\frac{2y^2 u^2}{\tilde{c}^2}\right) \int_0^1 dt \sinh(2t|y|) \exp(-\tilde{c}^2 t^2/2), \quad (24)$$

The first term in this expression shows that everywhere in the regime of "weak asymmetry" $\tilde{c} < \infty$ a finite fraction of eigenvalues remains on the real axis.

Such a behavior is qualitatively different from that typical for the case of "weak non-Hermiticity" $\tilde{a} < \infty$, where eigenvalues acquire a nonzero imaginary part with probability one.

In the limit $\tilde{c} \gg 1$ the portion of real eigenvalues behaves like \tilde{c}^{-1} . Remembering the normalization of the parameter v , Eq.(17), it is easy to see that for the case of $v = O(1)$ the number of real eigenvalues should scale as \sqrt{N} . The fact that of the order of $N^{1/2}$ eigenvalues of strongly asymmetric real matrices stays real was first found numerically by Sommers et al. [29,46], and proved by Edelman [48].

IV. GAUSSIAN ALMOST-HERMITIAN MATRICES: FROM WIGNER-DYSON TO GINIBRE EIGENVALUE STATISTICS.

In the present section we concentrate on the particular case of almost-Hermitian random matrices with i.i.d. entries $\hat{J} = \hat{H}_1 + iv\hat{H}_2$, where \hat{H}_1 and \hat{H}_2 are taken independently from the Gaussian Unitary Ensemble (GUE) of *Hermitian* matrices with the probability density $\mathcal{P}(\hat{X}) = Q_N^{-1} \exp\left(-N/2J_0^2 \text{Tr} \hat{X}^2\right)$, $\hat{X} = \hat{X}^\dagger$.

Let us now introduce a new parameter $\tau = (1 - v^2)/(1 + v^2)$ and choose the scale constant J_0^2 to be equal to $(1 + \tau)/2$, for the sake of convenience. The parameter τ controls the magnitude of correlation between J_{jk} and J_{kj} : $\langle J_{jk} J_{kj} \rangle = \tau/N$, hence the degree of non-Hermiticity. This is easily seen from the probability density function for our ensemble of the random matrices \hat{J} :

$$\mathcal{P}(\hat{J}) = C_N^{-1} \exp\left[-\frac{N}{(1 - \tau^2)} \text{Tr}(\hat{J}\hat{J}^\dagger - \tau \text{Re} \hat{J}^2)\right], \quad (25)$$

where $C_N = [\pi^2(1 - \tau^2)/N^2]^{N^2/2}$. All the J_{jk} have zero mean and variance $\langle |J_{jk}|^2 \rangle = 1/N$ and only J_{jk} and J_{kj} are pairwise correlated. If $\tau = 0$ all the J_{jk} are mutually independent and we have maximum non-Hermiticity. When τ approaches unity, J_{jk} and J_{kj}^* are related via $J_{jk} = J_{kj}^*$ and we are back to an ensemble of Hermitian matrices.

Our first goal is to determine the n -eigenvalue correlation functions in the ensemble of random matrices specified by Eq. (25). The density of the joint distribution of eigenvalues in the ensemble is given by

$$\mathcal{P}_N(Z_1, \dots, Z_N) = \frac{N^{N(N+1)/2}}{\pi^N 1! \dots N! (1 - \tau^2)^{N/2}} \exp\left\{\frac{-N}{1 - \tau^2} \sum_{j=1}^N \left[|Z_j|^2 - \frac{\tau}{2}(Z_j^2 + Z_j^{*2})\right]\right\} \prod_{j < k} |Z_j - Z_k|^2. \quad (26)$$

To derive Eq. (26) we integrate $\mathcal{P}(\hat{J})$ from Eq. (25) over the surface of all complex matrices whose eigenvalues are Z_1, \dots, Z_N . Following Dyson [44], p.501, see also [48]) we decompose every complex matrix with distinct eigenvalues as $\hat{J} = \hat{U}(\hat{Z} + \hat{R})\hat{U}^\dagger$, where $\hat{Z} = \text{diag}\{Z_1, \dots, Z_N\}$, \hat{U} is a unitary matrix, and \hat{R} is a strictly upper-triangular one. If we label the eigenvalues and require the first non-zero element in each column of \hat{U} to be positive, then the decomposition is unique. The Jacobian of the transformation $\hat{J} \rightarrow \{\hat{Z}, \hat{R}, \hat{U}\}$ depends only on \hat{Z} and is given by the squared modulus of the Vandermonde determinant. So, integrating out \hat{R} and \hat{U} is straightforward and the resulting expression is Eq. (26).

The form of the distribution Eq. (26) allows one to employ the powerful method of orthogonal polynomials [44]. Let $H_n(z)$ denote the n -th Hermite polynomial,

$$H_n(z) = \frac{(\pm i)^n}{\sqrt{2\pi}} \exp\left(\frac{z^2}{2}\right) \int_{-\infty}^{\infty} dt t^n \exp\left(-\frac{t^2}{2} \mp izt\right). \quad (27)$$

The crucial observation borrowed from the paper [69] (see also the related paper [70]) is that the polynomials

$$p_n(Z) = \frac{\tau^{n/2} \sqrt{N}}{\sqrt{\pi} \sqrt{n!} (1 - \tau^2)^{1/4}} H_n\left(\sqrt{\frac{N}{\tau}} Z\right), \quad (28)$$

$n = 0, 1, 2, \dots$, are orthogonal in the *complex plane* $Z = X + iY$ with the weight function

$$w^2(Z) = \exp\left\{-\frac{N}{(1 - \tau^2)} \left[|Z|^2 - \frac{\tau}{2}(Z^2 + Z^{*2})\right]\right\},$$

i. e. $\int d^2 Z p_n(Z) p_m(Z^*) w^2(Z) = \delta_{nm}$, where $d^2 Z = dX dY$. The standard machinery of the method of orthogonal polynomials [44] yields the n -eigenvalue correlation functions

$$R_n(Z_1, \dots, Z_n) = \frac{N!}{(N-n)!} \int d^2 Z_{n+1} \dots d^2 Z_N \mathcal{P}_N\{Z\} \quad (29)$$

in the form

$$R_n(Z_1, \dots, Z_n) = \det [K_N(Z_j, Z_k^*)]_{j,k=1}^n,$$

where the kernel $K_N(Z_1, Z_2^*)$ is given by

$$K_N(Z_1, Z_2^*) = w(Z_1) w(Z_2^*) \sum_{n=0}^{N-1} p_n(Z_1) p_n(Z_2^*). \quad (30)$$

With Eqs. (28)–(30) in hand, let us first examine the regime of strong non-Hermiticity, i.e. the case when $\lim_{N \rightarrow \infty} (1 - \tau) > 0$. In this regime the averaged density of eigenvalues $N^{-1} R_1(Z)$ is asymptotically zero outside the ellipse $[\text{Re} z / (1 + \tau)]^2 + [\text{Im} z / (1 - \tau)]^2 \leq 1$. Inside the ellipse $\lim_{N \rightarrow \infty} N^{-1} R_1(Z) = [\pi(1 - \tau^2)]^{-1}$. This sets a microscopic scale on which the averaged number of eigenvalues in any domain of unit area remains finite when $N \rightarrow \infty$. Remarkably, the τ -dependence is essentially trivial on this scale: the statistical properties of eigenvalues are described by $\tilde{R}_n(z_1, \dots, z_n) \equiv N^{-n} R_n(\sqrt{N} Z_1, \dots, \sqrt{N} Z_n)$ and

$$\lim_{N \rightarrow \infty} \tilde{R}_n(z_1, \dots, z_n) = \left[\frac{1}{\pi(1 - \tau^2)} \right]^n e^{-\frac{1}{1 - \tau^2} \sum_{j=1}^n |z_j|^2} \det \left[e^{\frac{1}{1 - \tau^2} z_j z_k^*} \right]_{j,k=1}^n. \quad (31)$$

This limiting relation can be inferred [50] from Mehler's formula for the Hermite polynomials [71]. After the trivial additional rescaling $z \rightarrow z\sqrt{1 - \tau^2}$ the expression on the right-hand side in Eq. (31) becomes identical to that found by Ginibre [43].

Now we move on to the regime of weak non-Hermiticity. We know that in this regime new non-trivial correlations occur on the scale: $\text{Im} Z_{1,2} = O(1/N)$, $\text{Re} Z_1 - \text{Re} Z_2 = O(1/N)$. Correspondingly, we introduce new variables x, y_1, y_2, ω in such a way that: $x = \text{Re}(Z_1 + Z_2)/2$, $y_{1,2} = N \text{Im}(Z_{1,2})$, $\omega = N \text{Re}(Z_1 - Z_2)$, and consider them finite when performing the limit $N \rightarrow \infty$.

Substituting Eq.(27) into Eq.(30) and using the above definitions we can explicitly perform the limit $N \rightarrow \infty$, taking into account that $\lim_{N \rightarrow \infty} N(1 - \tau) = \alpha^2/2$. The details of the procedure are given elsewhere [50]. In this regime

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N^2} K_N \left(x + \frac{\omega/2 + iy_1}{N}, x - \frac{\omega/2 + iy_2}{N} \right) \\ &= \frac{1}{\pi\alpha} \exp \left\{ -\frac{y_1^2 + y_2^2}{\alpha^2} + \frac{ix(y_1 - y_2)}{2} \right\} \int_{-\pi\nu_{sc}(x)}^{\pi\nu_{sc}(x)} \frac{du}{\sqrt{2\pi}} \exp \left[-\frac{\alpha^2 u^2}{2} - u(y_1 + y_2) + 2i\omega u \right], \end{aligned} \quad (32)$$

with $\nu_{sc}(X) = \frac{1}{2\pi} \sqrt{4 - X^2}$ standing for the Wigner semicircular density of real eigenvalues of the Hermitian part \hat{H} of the matrices \hat{J} .

Equation (32) constitutes the most important result of the present section. The kernel K_N given by Eq. (32) determines all the properties of complex eigenvalues in the regime of weak non-Hermiticity. For instance, the mean value of the density $\rho(Z) = \sum_{i=1}^N \delta^{(2)}(Z - Z_i)$ of complex eigenvalues $Z = X + iY$ is given by $\langle \rho(Z) \rangle = K_N(Z, Z^*)$. Putting $y_1 = y_2$ and $\omega = 0$ in Eqs (32) we immediately recover the density Eq.(22) found by the supersymmetry approach ².

One of the most informative statistical measures of the spectral correlations is the ‘connected’ part of the two-point correlation function of eigenvalue densities:

²In the present section we normalized \hat{H}_2 in such a way that for weak non-Hermiticity regime we have $\lim_{N \rightarrow \infty} \text{Tr} \hat{H}_2^2 = N$, whereas the normalization Eq.(17) gives $\lim_{N \rightarrow \infty} \text{Tr} \hat{H}_2^2 = N(1 + w^2)$. It is just because of this difference the parameter $\tilde{\alpha}$ entering Eq.(22) contains an extra factor $1 + w^2$ as compared to the present case.

$$\langle \rho(Z_1) \rho(Z_2) \rangle_c = \langle \rho(Z_1) \rangle \delta^{(2)}(Z_1 - Z_2) - \mathcal{Y}_2(Z_1, Z_2), \quad (33)$$

In particular, it determines the variance $\Sigma^2(D) = \langle n(D)^2 \rangle - \langle n(D) \rangle^2$ of the number $n = \int_D d^2 Z \rho(Z)$ of complex eigenvalues in any domain D in the complex plane as:

$$\Sigma_2(D) = \int_D d^2 Z_1 \int_D d^2 Z_2 [\langle \rho(Z_1) \rho(Z_2) \rangle - \langle \rho(Z_1) \rangle \langle \rho(Z_2) \rangle] = \int_D d^2 Z \langle \rho(Z) \rangle - \int_D d^2 Z_1 \int_D d^2 Z_2 \mathcal{Y}_2(Z_1, Z_2) \quad (34)$$

Comparing Eq.(33) with the definition Eqs. (28)–(30) we see that the *cluster function* $\mathcal{Y}_2(Z_1, Z_2)$ is expressed in terms of the kernel K_N as $\mathcal{Y}_2(Z_1, Z_2) = |K_N(Z_1, Z_2^*)|^2$.

It is evident that in the limit of weak non-Hermiticity the kernel K_N depends on X only via the semicircular density $\nu_{sc}(X)$. Thus, it does not change with X on the local scale comparable with the mean spacing along the real axis $\Delta \sim 1/N$.

The cluster function is given by the following explicit expression:

$$\mathcal{Y}(\omega, y_1, y_2) = \frac{N^4}{\pi^2 \alpha^2} e^{-2 \frac{y_1^2 + y_2^2}{\alpha^2}} \left| \int_{-\pi \nu(X)}^{\pi \nu(X)} \frac{du}{(2\pi)^{1/2}} \exp \left[-\frac{\alpha^2 u^2}{2} - u(y_1 + y_2) + i u \omega \right] \right|^2 \quad (35)$$

The parameter $a = \pi \nu(X) \alpha$ controls the deviation from Hermiticity³. When $a \rightarrow 0$ the cluster function tends to GUE form $\mathcal{Y}_2(\omega, y_1, y_2) = \frac{N^4}{\pi^2} \delta(y_1) \delta(y_2) \frac{\sin^2 \frac{\pi \nu(X) \omega}{\alpha^2}}{\omega^2}$. In the opposite case $a \gg 1$ the limits of integration in Eq.(35) can be effectively put to $\pm \infty$ due to the Gaussian cutoff of the integrand. The corresponding Gaussian integration is trivially performed yielding in the original variables Z_1, Z_2 the expression equivalent (up to a trivial rescaling) to that found by Ginibre [43]: $\mathcal{Y}_2(Z_1, Z_2) = (N^2/\pi \alpha^2)^2 \exp\{-N^2|Z_1 - Z_2|^2/\alpha^2\}$.

The operation of calculating the Fourier transform of the cluster function over its arguments ω, y_1, y_2 amounts to simple Gaussian and exponential integrations. Performing them one finds the following expression for the *spectral form-factor*:

$$\begin{aligned} b(q_1, q_2, k) &= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \mathcal{Y}_2(\omega, y_1, y_2) \exp\{2\pi i(\omega k + y_1 q_1 + y_2 q_2)\} \\ &= N^4 \exp\left\{-\frac{\alpha^2}{2}(q_1^2 + q_2^2 + 2k^2)\right\} \frac{\sin\left[\frac{\pi^2 \alpha^2 (q_1 + q_2)(\nu(X) - |k|)}{\pi^2 \alpha^2 (q_1 + q_2)}\right]}{\pi^2 \alpha^2 (q_1 + q_2)} \theta(\nu(X) - |k|) \end{aligned} \quad (36)$$

where $\theta(u) = 1$ for $u > 0$ and zero otherwise.

We see, that everywhere in the regime of weak non-Hermiticity $0 < \alpha < \infty$ the formfactor shows a kink-like behavior at $|k| = \nu(X)$. This feature is inherited from the corresponding Hermitian counterpart—the Gaussian Unitary Ensemble. It reflects the oscillations of the cluster function with ω which is a manifestation of the long-ranged order in eigenvalue positions along the real axis [2]. When non-Hermiticity increases the oscillations become more and more damped.

As we already discussed above the knowledge of the formfactor allows one to determine the variance Σ_2 of a number of eigenvalues in any domain D of the complex plane. Small Σ_2 is a signature of a tendency for levels to form a crystal-like structure with long correlations. In contrast, increase in the number variance signals about growing decorrelations of eigenvalues.

In a general case this expression is not very transparent, however. For this reason we restrict ourselves to the simplest case, choosing the domain D to be the infinite strip of width L_x (in units of mean spacing along the real axis $\Delta = (\nu_{sc}(0)N)^{-1}$) oriented perpendicular to the real axis: $0 < \text{Re} Z < L_x \Delta$; $-\infty < \text{Im} Z < \infty$. Such a choice means that we look only at real parts of complex eigenvalues irrespective of their imaginary parts. It is motivated, in particular, by the reasons of comparison with the GUE case, for which the function $\Sigma(L_x)$ behaves at large L_x logarithmically: $\Sigma(L_x) \propto \ln L_x$ [2].

After simple calculations one finds⁴

$$\Sigma_2(L_x) = L_x \left[1 - \frac{2}{\pi^2} \int_0^{L_x} \frac{dk}{k^2} \left(1 - \frac{k}{L_x}\right) \sin^2(\pi k) e^{-\left(\frac{\alpha k}{L_x}\right)^2} \right] \quad (37)$$

³ In our earlier Letter [39] we used the definition of the parameter a different by a factor of 2 from the present one.

⁴In our earlier Letter [39] the expression Eq.(37) and formulae derived from it erroneously contained πa instead of a .

First of all, it is evident that Σ_2 grows systematically with increase in the degree of non-Hermiticity $a = \pi\nu(0)\alpha$. This fact signals on the gradual decorrelation of the *real* parts $\text{Re}Z_i$ of complex eigenvalues. It can be easily understood because of increasing possibility for eigenvalues to avoid one another along the $Y = \text{Im}Z$ direction, making their projections on the real axis X to be more independent.

In order to study the difference from the Hermitian case in more detail let us consider again the large L_x behavior. In that case the upper limit of the integral in Eq.(37) can be set to infinity. Then it is evident, that the number variance is only slightly modified by non-Hermiticity as long as $a \ll L_x$. We therefore consider the case $a \gg 1$ when we expect essential differences from the Hermitian case.

In a large domain $1 \ll L_x \sim a$ the second term in the integrand of Eq.(37) can be neglected and the number variance grows like $\Sigma(L_x) = L_x f(L_x/a)$. We find it more transparent to rewrite the function $f(u)$ in an equivalent form:

$$f(u) = 1 + \frac{2}{\sqrt{\pi}} \left\{ \frac{1}{2\pi u} \left(1 - e^{-\pi^2 u^2} \right) - \int_0^{\pi u} dt e^{-t^2} \right\}.$$

which can be obtained from Eq.(37) after a simple transformation.

For $u = L_x/a \ll 1$ we have simply $f \approx 1$ and hence a linear growth of the number variance. For $u \gg 1$ we have $f \approx (\pi^{3/2}u)^{-1}$. Thus, $\Sigma_2(L_x)$ slows down: $\Sigma_2(L_x) \approx \frac{a}{(\pi)^{3/2}}$.

Only for exponentially large L_x such that $\ln(L_x/a) \sim a$ second term in Eq.(37) contributes significantly. Calculating its contribution explicitly and remembering that $\Sigma_2^{(1)}|_{L_x \gg a} \approx a/(\pi^{3/2})$ we finally find:

$$\Sigma_2(L_x \gg a) = \frac{a}{\pi^{3/2}} + \frac{1}{\pi^2} \left(\ln \left(\frac{L_x}{a} \right) - \frac{\gamma}{2} \right)$$

where γ is Euler's constant. This logarithmic growth of the number variance is reminiscent of that typical for real eigenvalues of the Hermitian matrices.

Another important spectral characteristics which can be simply expressed in terms of the cluster function is the small-distance behavior of the nearest neighbor distance distribution [44,2,40].

We define the quantity $p(Z_0, S)$ as the probability density of the following event: i) There is exactly one eigenvalue at the point $Z = Z_0$ of the complex plane. ii) Simultaneously, there is exactly one eigenvalue on the circumference of the circle $|Z - Z_0| = S$ iii) All other eigenvalues Z_i are *out* of that circle: $|Z_i - Z_0| > S$.

As a consequence, the normalization condition is: $\int d^2 Z_0 \int_0^\infty dS p(Z_0, S) = 1$. In particular, for Hermitian matrices with real eigenvalues one has the relation: $p(Z_0, S) = \delta(\text{Im}Z_0) \tilde{p}_X(S)$, with $\tilde{p}_X(S)$ being the conventional "nearest neighbor spacing" distribution at the point X of the real axis [44].

We are interested in finding the leading small- S behavior for the function $p(Z_0, S)$. It turns out to be given by the following expression [50]:

$$p(Z_0, S) \approx \frac{S}{N} \int_0^{2\pi} d\theta [\langle \rho(Z_0) \rangle \langle \rho(Z_0 + S e^{i\theta}) \rangle - \mathcal{Y}_2(Z_0, Z_0 + S e^{i\theta})] \quad (38)$$

where we used the definition of the cluster function, Eq.(33).

In the regime of weak non-Hermiticity this formula is valid as long as the parameter S is small in comparison with a typical separation between real eigenvalues of the Hermitian counterpart: $S \ll \Delta \sim 1/N$.

Substituting the expression Eqs.(22,35) for the mean density and the cluster function into Eq.(38) one arrives after a simple algebra to the probability density to have one eigenvalue at the point $Z_0 = X + iy_0\Delta$ and its closest neighbor at the distance $|z_1 - z_0| = s\Delta$, $\Delta = (\nu(X)N)^{-1}$, such that $s \ll 1$:

$$p_\alpha(X + iy_0\Delta, s\Delta)|_{s \ll 1} = \frac{\pi\nu^2(X)}{2} \left[g_a(y_0) \frac{\partial^2}{\partial y_0^2} g_a(y_0) - \left(\frac{\partial}{\partial y_0} g_a(y_0) \right)^2 \right] e^{-4\frac{y_0^2}{a^2} \frac{s^3}{a^2}} \int_0^\pi d\theta \exp \left[-\frac{2}{a^2} (s^2 \cos^2 \theta - 2y_0 s \cos \theta) \right]$$

where

$$g_a(y) = \int_{-1}^1 \frac{du}{(2\pi)^{1/2}} \exp \left\{ -\frac{a^2 u^2}{2} - 2uy \right\}$$

First of all it is easy to see that in the limit $a \gg 1$ one has: $p_{a \gg 1}(Z_0, s \ll 1) = \frac{2}{\pi} (s/a^2)^3$ in agreement with the cubic repulsion generic for strongly non-Hermitian random matrices [43,26,40]. On the other hand one can satisfy oneself that in the limit $a \rightarrow 0$ we are back to the familiar GUE quadratic level repulsion: $p_{a \rightarrow 0}(Z_0, s \ll 1) \propto \delta(y_0) s^2$.

In general, the expression Eq.(39) describes a smooth crossover between the two regimes, although for any $a \neq 0$ the repulsion is always cubic for $s \rightarrow 0$.

To this end, an interesting situation may occur when deviations from the Hermiticity are very weak: $a \ll \sqrt{2}$ and ‘observation points’ Z_0 are situated sufficiently far from the real axis: $2|y_0|/a \gg 2^{-1/2}$.

Under this condition the following three regions for the parameter s should be distinguished: i) $\frac{s}{a} \ll \frac{a}{4|y_0|}$ ii) $\frac{a}{4|y_0|} \ll \frac{s}{a} \ll 2\frac{|y_0|}{a}$ and finally iii) $2^{-1/2} \ll 2\frac{|y_0|}{a} \ll \frac{s}{a} \ll a^{-1}$.

In the regimes i) and ii) the term linear in $\cos\theta$ in the exponent of Eq.(39) dominates yielding the result of integration to be the modified Bessel function $\pi I_0\left(\frac{4y_0 s}{a^2}\right)$. In the regime iii) the term quadratic in $\cos\theta$ dominates producing $2\pi e^{-(s/a)^2} I_0[(s/a)^2] \approx (2\pi a/s)^{1/2}$. As the result, the distribution $p(Z_0, s)$ displays the following behavior:

$$p_\alpha(Z_0, s) = \frac{\pi^2 \nu^2}{2} \left[g_0(y_0) \frac{\partial^2}{\partial y_0^2} g_0(y_0) - \left(\frac{\partial}{\partial y_0} g_0(y_0) \right)^2 \right] e^{-4\frac{y_0^2}{a^2}} \begin{cases} \frac{s^3}{a^2} & \text{for } \frac{s}{a} \ll \frac{a}{4|y_0|} \\ \frac{s^{5/2}}{2a\sqrt{2\pi|y_0|}} & \text{for } \frac{a}{4|y_0|} \ll \frac{s}{a} \ll 2\frac{|y_0|}{a} \\ \sqrt{\frac{2}{\pi}} \frac{s^2}{a} & \text{for } 2\frac{|y_0|}{a} \ll \frac{s}{a} \ll a^{-1} \end{cases},$$

with $g_0(y) \equiv g_a(y)|_{a=0}$.

Unfortunately, it might be a very difficult task to detect numerically the unusual power law $p(s) \propto s^{5/2}$ because of the low density of complex eigenvalues in the observation points reflected by the presence of the Gaussian factor in the expression Eq.(39).

V. CONCLUSION

In the present paper we addressed the issue of eigenvalue statistics of large weakly non-Hermitian matrices.

Our original motivation came from the field of resonance chaotic scattering. The resonances, which are complex poles of the scattering matrix enter the theory as complex eigenvalues of a non-Hermitian effective Hamiltonian of a particular type: $\mathcal{H}_{ef} = \hat{H} - i\hat{\Gamma}$. We demonstrated that one can extract mean density of such poles employing a mapping onto the supermatrix non-linear σ -model. We also have shown how the resolvent of the non-Hermitian Hamiltonian \mathcal{H}_{ef} can be used to describe the process of chaotic photodissociation and presented the crossection autocorrelation function.

Guided by our experience with the resonances, we found a regime of weak non-Hermiticity for other types of non-selfadjoint random matrices. The regime can be defined as that for which the imaginary part $\text{Im}Z$ of a typical complex eigenvalue is of the same order as the mean eigenvalue separation Δ for the corresponding Hermitian counterpart.

Exploiting a mapping to the non-linear σ -model we are able to show that there are three different ‘‘pure’’ symmetry classes of weakly non-Hermitian matrices: i) almost Hermitian with complex entries ii) almost symmetric with real entries and iii) complex symmetric ones. Within each of these classes the eigenvalue statistics is *universal* in a sense that it is the same irrespective of the particular distribution of matrix entries up to an appropriate rescaling. There are also crossover regimes between all three classes.

Our demonstration of universality was done explicitly for the density of complex eigenvalues of matrices with independent entries. Within the non-linear σ -model formalism one can easily provide a heuristic proof of such a universality for higher correlation functions as well as for ‘‘rotationally invariant’’ matrix ensembles, see [68]. The above feature is a great advantage of the supersymmetry technique.

A weak point of that method is a very complicated representation of the ensuing quantities. It seems, that the explicit evaluation of the higher correlation functions is beyond our reach at the moment, and even a calculation of the mean density requires a lot of effort, see [38,32]. As a result, at present time the mean density is known explicitly only for the cases i) and ii).

Fortunately, because of the mentioned universality another strategy can be pursued. Namely, one can concentrate on the particular case of matrices with independent, Gaussian distributed entries for which alternative analytical techniques might be available. Such a strategy turned out to be a success for the simplest case of complex almost-Hermitian matrices, where we found the problem to be an exactly soluble one by the method of orthogonal polynomials. This fact allowed us to extract all the correlation functions in a mathematically rigorous way [39,50].

One might hope that combining the supersymmetric method and the method of orthogonal polynomials one will be able to elevate our understanding of properties of almost-Hermitian random matrices to the level typical for their Hermitian counterparts.

From this point of view a detailed numerical investigation of different types of almost-Hermitian random matrices is highly desirable. Recently, an interesting work in this direction appeared motivated again by the theory of chaotic

scattering [10]. Unfortunately, matrices \mathcal{H}_{ef} emerging in that theory are different from the Gaussian matrices because of the specific form of the antihermitean perturbation $i\hat{\Gamma}$ necessary to ensure the unitarity of the scattering matrix. This fact makes impossible a quantitative comparison of our results with those obtained in [10]. The qualitative fact of increase in number variance with increase in non-Hermiticity agrees well with our findings.

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